PROBABILITY AND STATISTICS 550.310 SPRING 2009
FINAL EXAM

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No document. No calculator. The exam is worth 110 points.

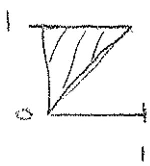
1. Multiple Choices (30 points)

Clearly circle the letter of the most correct answer to each question.

(1) If A and B are independent then
  (a) $P(A \cup B) = P(A)P(B)$
  (b) $P(A \cup B) = P(A) + P(B)$
  (c) $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

(2) If $P(B) > 0$ then $P(A|B) =$
  (a) $\frac{P(A \cap B)}{P(B)}$
  (b) $\frac{P(A \cap B)}{P(A)}$
  (c) $P(A \cap B)P(B)$

(3) If X and Y are continuous random variables with Uniform distribution over the triangle $\{(x, y), 0 < x < 1, 0 < y < 1, x < y\}$ then X and Y are independent
  (a) true
  (b) false



(4) For any random variables X and Y, if X and Y are independent then $E(XY) = E(X)E(Y)$
  (a) true
  (b) false

(5) For any random variable X, $E(X^2) \geq E^2(X)$.
  (a) true
  (b) false

(6) If X and Y are independent and both Uniform over the interval $[0, 1]$ then $X + Y$ is Uniform over the Interval $[0, 2]$
  (a) true
  (b) false

Date: February 20th, 2009.

2. Problems

Show all work to obtain credits. For each single question, simplify until obtaining an irreducible fraction.

2.1. **Independence and Correlation (20 points).** Let X and Y be 2 discrete random variables. The range of X is $R_X = \{0, 1, 2\}$, while the range of Y is $R_Y = \{1, 2, 3\}$. Their joint point mass function f is given in the table below:

x \ y	1	2	3	
0	0	.25	0	1/4
1	.25	0	.25	1/2
2	0	.25	0	1/4
	1/4	1/2	1/4	

(1) Compute $E[X]$, $V[X]$, $E[Y]$, $V[Y]$

$$E[X] = 0(1/4) + 1(1/2) + 2(1/4) = 1$$

$$E[X^2] = 0^2\left(\tfrac{1}{4}\right) + 1^2\left(\tfrac{1}{2}\right) + 2^2\left(\tfrac{1}{4}\right) = \tfrac{3}{2}$$

$$V[X] = \tfrac{3}{2} - 1^2 = \tfrac{1}{2}$$

$$E[Y] = 1\left(\tfrac{1}{4}\right) + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{4}\right) = 2$$

$$E[Y^2] = 1^2\left(\tfrac{1}{4}\right) + 2^2\left(\tfrac{1}{2}\right) + 3^2\left(\tfrac{1}{4}\right) = \tfrac{1}{4} + 2 + \tfrac{9}{4} = \tfrac{18}{4} = \tfrac{9}{2}$$

(2) Compute $E[XY]$, $Cov(X, Y)$

$$V[Y] = \tfrac{9}{2} - 2^2 = \tfrac{1}{2}$$

$$E[XY] = 1\left(\tfrac{1}{4}\right) + 3\left(\tfrac{1}{4}\right) + 4\left(\tfrac{1}{4}\right) = 2$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 0$$

(3) Are X and Y independent ? Justify your answer to receive credits.

$$P(X=2, Y=1) = 0$$

$$P(X=2)P(Y=1) = \tfrac{1}{4}\,\tfrac{1}{4} = \tfrac{1}{16} \neq 0$$

hence X and Y are not independent

2.2. **Confidence Interval (20 points).** for this exercise, you will use the values for the cumulative distribution function of the standard Normal distribution given in the table below

z	0	1	1.65	1.96	2	3
$\phi(z)$	.5	.84	.95	.975	.98	.999

Let $X_1, \ldots, X_{16}$ be a random sample (i.i.d random variables) with common distribution the Normal distribution $N(\mu, \sigma^2)$

(1) Assume that $\mu = 50$, $\sigma^2 = 4$ and let

$$\bar{X} = \frac{1}{16}\sum_{i=1}^{16} X_i, \qquad E[\bar{X}] = 50$$

Compute $P(49 \leq \bar{X} \leq 51)$. Do all computations. $V[\bar{X}] = \frac{4}{16} = \frac{1}{4}$

$P(49 \leq \bar{X} \leq 51)$

$= P\left(2(49-50) \leq 2(\bar{X}-50) \leq 2(51-50)\right)$

$= P(-2 \leq Z \leq 2)$ where $Z \sim N(0,1)$

$= \phi(2) - \phi(-2) = \phi(2) - [1-\phi(2)] = 2\phi(2) - 1 = 2(.98) - 1 = 96\%$

(2) Assume now that μ is unknown and as previously $\sigma^2 = 4$. We observe $\bar{x} = 51$. Propose a 95% two-sided confidence interval for μ.

$\bar{x} \pm z\left(\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}}$

$51 \pm z(.025)\frac{2}{4}$

$51 \pm \frac{1.96}{2} = 51 \pm .98 = [50.02; 51.98]$

2.3. **Change of distribution (20 points).** Let X be a continuous random variable with distribution Uniform($[0;1]$). Let

$$Y = -\frac{1}{\lambda}\ln(1-X)$$

where $\lambda > 0$ is a constant.

(1) Compute the range of Y. Compute the cumulative distribution of Y.

$$0 < X < 1 \Rightarrow 0 < 1-X < 1 \Rightarrow -\infty < \ln(1-X) < 0$$

$0 < t < +\infty$ $$\Rightarrow 0 < -\frac{1}{\lambda}\ln(1-X) < +\infty$$

$$P(Y \leqslant t) = P\left(-\frac{1}{\lambda}\ln(1-X) \leqslant t\right)$$

$$= P(\ln(1-X) \geqslant -\lambda t) = P(1-X \geqslant e^{-\lambda t})$$

$$= P(X \leqslant 1 - e^{-\lambda t}) = 1 - e^{-\lambda t}$$

(2) Compute the probability density of Y.
Hint: This distribution is in your formulae pages.

$t > 0$;

$$f_Y(t) = \frac{\partial}{\partial t}F_Y(t) = \frac{\partial}{\partial t}\left[1 - e^{-\lambda t}\right]$$

$$= \lambda e^{-\lambda t}$$

$Y \sim Exp(\lambda)$

2.4. **Books(10 points).** A bookstore buys 4 copies of a book for \$10 each. Each copy sells for \$15. The number of copies sold during a three month period, notated Y, is random with Binomial distribution $Binomial(n = 4, p = \frac{1}{2})$. At the end of the 3 months period, each unsold book is redeemed at \$5.

(1) Compute the Expected value of the net profit generated by this book. Show all work. Do all computations.

the net profit

$$X = -40 + 15Y + (4-Y)5$$

$$= 10Y - 20$$

$$E[X] = 10E[Y] - 20$$

$$E[Y] = 2\ ;\ E[X] = 0$$

2.5. **Urn (10 points).** An urn contains 3 blue balls, 5 green and 2 reds. Pick 3 balls, randomly, one by one, without replacement from the urn.

(1) Compute the probability that the second as well as the third are red. Shaw all work. Do all computations

We write $RR =$ " 2^{nd} and 3^{rd} or red "

$B =$ " first is blue "

$G =$ " first is green "

$R =$ " first is red "

$$P(RR) = P(RR|B)P(B) + P(RR|G)P(G) + \underbrace{P(RR|R)}_{0}P(R)$$

$$P(B) = \frac{3}{10}; \quad P(G) = \frac{5}{10}$$

$$R(RR|B) = P(RR|G) = \frac{\cancel{2}}{9}\frac{1}{\cancel{8}4} = \frac{1}{36}$$

$$P(RR) = \frac{1}{36}\frac{3}{10} + \frac{1}{36}\frac{5}{10} = \frac{1}{\cancel{36}\,9}\frac{\cancel{8}^{4}}{\cancel{10}5} = \frac{1}{45}$$