550.310

Hw 8 solution.

5.1. $E[X] = \mu$, $Var[X] = \sigma^2$

$$E[\frac{X-\mu}{\sigma}] = \frac{1}{\sigma}E[X] - \frac{\mu}{\sigma} = \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0$$

$$Var[\frac{X-\mu}{\sigma}] = \frac{1}{\sigma^2}Var[X] = \frac{\sigma^2}{\sigma^2} = 1$$

$\Rightarrow$ standard deviation of $\frac{X-\mu}{\sigma}$ is $\sqrt{Var[\frac{X-\mu}{\sigma}]} = 1$

5.2. $X \sim Unif(0,1)$ $E[X] = \frac{1}{2}$ $f(x) = 1$

a) $Var[X] = E[(X - E[X])^2]$

$$= \int_0^1 (x-\tfrac{1}{2})^2 \cdot 1 \, dx$$

$$= \int_0^1 (x^2 - x + \tfrac{1}{4})\, dx$$

$$= (\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x)\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{4}$$

$$= \frac{1}{12}$$

b) $Y \sim Unif(0,3)$; then $Y = 3X$

$$Var[Y] = Var[3X] = 9Var[X] = \frac{9}{12} = \frac{3}{4}$$

c) $Z \sim Unif(-1.5, 1.5)$; then $Z = Y - 1.5$, $Var[Z] = Var[Y-1.5] = Var[Y] = \frac{3}{4}$

27. $f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$h(X, Y) = |X - Y|$

$$E(|X-Y|) = \int_0^1 \int_0^1 |x-y| \cdot 3x^2 \cdot 2y \, dy \, dx$$

$$= \int_0^1 \int_0^x (x-y) \cdot 6x^2 y \, dy \, dx + \int_0^1 \int_x^1 (y-x) \cdot 6x^2 y \, dy \, dx$$

$$= \int_0^1 (3x^3y^2 - 2x^2y^3) \Big|_{y=0}^{x} dx + \int_0^1 (2x^2y^3 - 3x^3y^2) \Big|_{y=x}^{1} dx$$

$$= \int_0^1 x^5 \, dx + \int_0^1 (x^5 - 3x^3 + 2x^2) \, dx$$

$$= \frac{1}{6}x^6 \Big|_0^1 + \left(\frac{1}{6}x^6 - \frac{3}{4}x^4 + \frac{2}{3}x^3\right)\Big|_0^1$$

$$= \frac{1}{6} + \frac{1}{6} - \frac{3}{4} + \frac{2}{3}$$

$$= \frac{1}{4}$$

Extra question: $Z \sim N(0,1)$

$Y = e^Z$

$$F_Y(y) = P(Y \le y) = P(e^Z \le y) = P(Z \le \ln y) = F_Z(\ln y)$$

$$f_Y(y) = \frac{d F_Y(y)}{dy} = \frac{d F_Z(\ln y)}{dy} \underset{\downarrow}{=} f_Z(\ln y) \cdot \frac{d(\ln y)}{dy} = f_Z(\ln y) \cdot \frac{1}{y} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}} \cdot \frac{1}{y}$$

(by chain rule)

30.

$p(x,y)$		Y: 0	5	10	15	
	0	.02	.06	.02	.10	.20
X	5	.04	.15	.20	.10	.49
	10	.01	.15	.14	.01	.31
		.07	.36	.36	.21	

$E[X] = \sum x \cdot p(x) = 0 \times (.20) + 5 \times (.49) + 10 \times (.31) = 5.55$

$E[Y] = \sum y \cdot p(y) = 0 \times (.07) + 5 \times (.36) + 10 \times (.36) + 15 \times (.21)$

$= 8.55$

a) $Cov(X,Y) = E[(X-E[X])(Y-E(Y))]$

$= \sum\sum (x-5.55)(y-8.55) \cdot p(x,y)$

$= -3.2025$

b) $E[X^2] = \sum x^2 \cdot p(x) = 43.25$

$E[Y^2] = \sum y^2 \cdot p(y) = 92.25$

$Var[X] = E[X^2] - (E[X])^2 = 12.4475$

$Var[Y] = E[Y^2] - (E[Y])^2 = 19.1475$

$\rho = \dfrac{Cov(X,Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}} \approx -0.2074$