550.310

HW 7 - solution.

3.7. $X \sim \text{Unif}[0,5]$

$Y = X(5-X) = 5X - X^2$

$E[Y] = E[5X - X^2] = \int_0^5 (5x - x^2)\frac{1}{5}\,dx = \frac{1}{2}x^2 - \frac{1}{5}\times\frac{1}{3}x^3\Big|_0^5 = \frac{25}{2} - \frac{125}{5\times 3}$

$= \boxed{\frac{25}{6} \text{ square inches}}$

3.8. $X \sim N(20, 5^2)$

Area $Y = \pi X^2$

$E[Y] = E[\pi X^2]$

suppose $Z = \frac{X-20}{5}$, we know $Z \sim N(0,1)$, thus $E(Z)=0$, $E(Z^2)=1$

$X = 5Z + 20$

so $E[Y] = E[\pi X^2] = E[\pi(5Z+20)^2] = \pi \cdot E[25Z^2 + 200Z + 400]$

$= \pi \cdot [25\underbrace{E[Z^2]}_{=1} + 200\underbrace{E[Z]}_{=0} + 400]$

$= \boxed{425\pi}$

3.9 $X \sim \text{Poisson}(\lambda)$ $\quad P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$ $\quad k = 0, 1, 2, \dots$

$$E\left(\frac{1}{X+1}\right) = \sum_{k\geq 0} \frac{1}{k+1} e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k\geq 0} e^{-\lambda} \frac{\lambda^k}{(k+1)!}$$

$$= e^{-\lambda} \cdot \frac{1}{\lambda} \cdot \sum_{k\geq 0} \frac{\lambda^{k+1}}{(k+1)!}$$

$$= e^{-\lambda} \cdot \frac{1}{\lambda} \cdot \sum_{j=1}^{\infty} \frac{\lambda^j}{j!}$$

$$= e^{-\lambda} \cdot \frac{1}{\lambda} \cdot \left[\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} - 1\right]$$

$$= e^{-\lambda} \cdot \frac{1}{\lambda} \cdot [e^{\lambda} - 1]$$

$$= \frac{1}{\lambda}[1 - e^{-\lambda}]$$

3.10. $X \sim \text{Unif}(1,2)$ $\quad f(x) = \begin{cases} 1 & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$

$$E\left[\frac{1}{X}\right] = \int_1^2 \frac{1}{x}\,dx = \ln 2 = 0.69 > \frac{1}{E[X]} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

4.1 X = the largest value

Y = the smallest value

Y\X	1	2	3	4	5	6	
1	1	2	2	2	2	2	11
2	0	1	2	2	2	2	9
3	0	0	1	2	2	2	7
4	0	0	0	1	2	2	5
5	0	0	0	0	1	2	3
6	0	0	0	0	0	1	1
	1	3	5	7	9	11	

a). $p_{X,Y}(x,y) = \begin{cases} 0 & x<y \\ \frac{1}{36} & x=y \\ \frac{2}{36}=\frac{1}{18} & x>y \end{cases}$

b) The marginal mass functions of X and Y are

$p_X(x)$ is, for $x=1,2,3,4,5,6$, respectively, $\frac{1}{36}, \frac{3}{36}, \frac{5}{36}, \frac{7}{36}, \frac{9}{36}, \frac{11}{36}$

$p_Y(y)$ is for $y=1,2,3,4,5,6$, respectively, $\frac{11}{36}, \frac{9}{36}, \frac{7}{36}, \frac{5}{36}, \frac{3}{36}, \frac{1}{36}$

c) They are not independent.

In particular, $P(X=1, Y=1) = \frac{1}{36} \neq$

$P(X=1) \cdot P(Y=1) = \frac{1}{36} \times \frac{11}{36}$

4.2 a) $\int_{y=0}^{+\infty} \int_{x=-y}^{y} (y^2 - x^2) e^{-y} \, dx \, dy$

$= \int_{y=0}^{+\infty} [y^2 e^{-y} x - \frac{1}{3} x^3 e^{-y}] \Big|_{-y}^{y} \, dy$

$= \int_{y=0}^{\infty} (y^3 e^{-y} - \frac{1}{3} y^3 e^{-y} - (-y^3 e^{-y} + \frac{1}{3} y^3 e^{-y})) \, dy$ $\quad 2 - \frac{2}{3} = \frac{4}{3}$

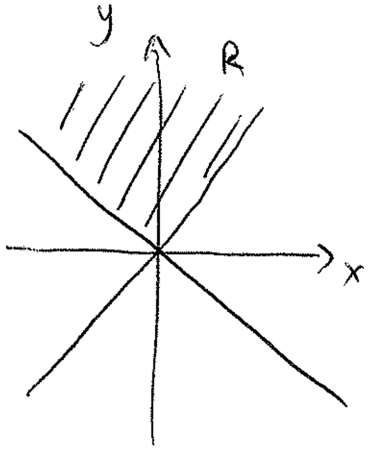
$= \int_{y=0}^{\infty} \frac{4}{3} y^3 e^{-y} \, dy = \frac{4}{3} \Gamma(4) \cdot \underbrace{\int_{y=0}^{+\infty} \frac{1}{\Gamma(4)} \cdot y^3 \cdot e^{-y} \, dy}_{\text{density of Gamma } (\alpha=4,\ \beta=1)} = \frac{4}{3}\Gamma(4) = \frac{4}{3} \times 3 \times 2 = 8$

hence $\boxed{c = \frac{1}{8}}$

b) $f_Y(y) = \int_{x=-\infty}^{+\infty} f(x,y)\,dx = \int_{-y}^{y} \frac{1}{8}(y^2-x^2)e^{-y}\,dx = \boxed{\frac{1}{\Gamma(4)} y^3 e^{-y}}, \quad y>0$

$Y \sim \text{Gamma}(4,1)$

c).



$R = \{x<0,\ y \geq -x\} \cup \{x>0,\ y \geq x\}$

$f_X(x) = \int_{y=-\infty}^{+\infty} f(x,y)\,dy$

$x>0,\ f_X(x) = \int_x^{+\infty} \frac{1}{8}(y^2-x^2)e^{-y}\,dy =$

$$\frac{\partial}{\partial y}\left[(ay^2+by+c)e^{-y}\right] = (2ay+b)e^{-y} - (ay^2+by+c)e^{-y}$$

$$= [-ay^2 + (2a-b)y + (b-c)]e^{-y}$$

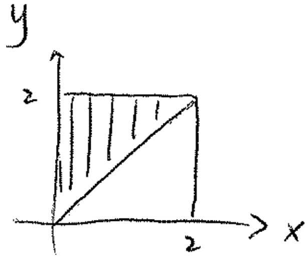
$$\begin{cases} -a = 1 \\ 2a-b=0 \\ b-c=-x^2 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=-2 \\ c=x^2-2 \end{cases}$$

$\boxed{x>0}$, $f_X(x) = \frac{1}{8}\left[(-y^2-2y+x^2-2)e^{-y}\right]\Big|_x^{+\infty} = \frac{-1}{8}(-\cancel{x^2}-2x+\cancel{x^2}-2)\cdot e^{-x}$

$$= \boxed{\frac{1}{4}(x+1)e^{-x}} = g(x)$$

$\boxed{x<0}$, $f_X(x) = \int_{-x}^{+\infty} f(x,y)\,dy = g(-x) = \boxed{\frac{1}{4}(1-x)\cdot e^{x}}$

4.3 $f(x,y) = \frac{1}{4}, \quad 0 \le x \le 2, \quad 0 \le y \le 2$



$P(Y > X) = \int_R f(x,y)\, dx\, dy = \frac{1}{4} \times 2 = \boxed{\frac{1}{2}}$

$R = \{ y > x, \ 0 \le x \le 2, \ 0 \le y \le 2 \}$

4.4 a) $f_X(x) = \int_{y=0}^{2} \frac{6}{7}(x^2 + \frac{x}{2}y)\, dy = \frac{6}{7}[x^2 y + \frac{x}{4}y^2]_0^2 = \boxed{\frac{6}{7}(2x^2 + x)}$

$0 < x < 1$

$f_Y(y) = \int_{x=0}^{1} \frac{6}{7}(x^2 + \frac{x}{2}y)\, dx = \frac{6}{7}[\frac{1}{3}x^3 + \frac{1}{4}yx^2]_0^1 = \boxed{\frac{2}{7} + \frac{3}{14}y}$ $\quad 0 < y < 2$

b) $f(x,y) \neq f_X(x) \cdot f_Y(y)$

so X and Y are dependent.

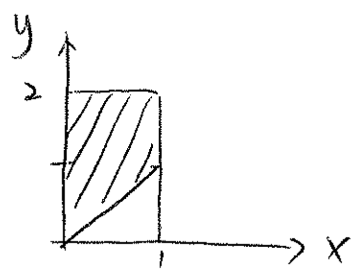
c) $P(X < \frac{1}{2}, Y < \frac{1}{2}) = \int_0^{1/2} \int_0^{1/2} \frac{6}{7}(x^2 + \frac{xy}{2})\, dy\, dx$

$= \int_0^{1/2} \frac{6}{7}[x^2 y + \frac{x}{4}y^2]_0^{1/2}\, dx$

$= \int_0^{1/2} \frac{6}{7}[\frac{1}{2}x^2 + \frac{1}{16}x]\, dx$

$= \left(\frac{6}{7} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot x^3 + \frac{6}{7} \cdot \frac{1}{16} \cdot \frac{1}{2} \cdot x^2\right)\Big|_0^{1/2} = \boxed{\frac{11}{448}}$

d) $P(Y > X) = \int_0^1 \int_x^2 \frac{6}{7}(x^2 + \frac{xy}{2})\, dy\, dx$



$= \int_0^1 \frac{6}{7}(x^2 y + \frac{x}{4}y^2)\Big|_x^2\, dx$

$= \int_0^1 \frac{6}{7}(2x^2 + x - \frac{5}{4}x^3)\, dx = \boxed{\frac{41}{56}}$

4.5 a) $f_X(x) = \int_{y=0}^{+\infty} e^{-(x+y)}\, dy$

$= e^{-x} \cdot (-1) \cdot e^{-y} \Big|_0^{+\infty}$

$= e^{-x}$

by symmetry, $f_Y(y) = e^{-y}$

$X \sim$ Exponential (1) , $Y \sim$ Exponential (1)

b) $f(x,y) = f_X(x) \cdot f_Y(y)$

so X and Y are independent

c) $P(Y > x) = \int_0^{+\infty} \int_x^{+\infty} f(x,y)\, dy\, dx$

$= \int_0^{+\infty} \int_x^{+\infty} e^{-(x+y)}\, dy\, dx$

$= \int_0^{+\infty} -e^{-(x+y)} \Big|_x^{+\infty} dx$

$= \int_0^{+\infty} e^{-2x}\, dx$

$= -\frac{1}{2} \cdot e^{-2x} \Big|_0^{+\infty} = \boxed{\frac{1}{2}}$