550.310

HW 6 - solution

2.1) see HW 5

2.2 $X \sim \text{Exponential}(\lambda)$, $\lambda = .01386$

the cdf is $F(x) = 1 - e^{-\lambda x}$; $x > 0$

① $P(X \le 100) = 1 - e^{-100\lambda} \approx .75$

② $P(X \le 200) = 1 - e^{-200\lambda} \approx .9375$

③ $P(100 < X < 200) = P(X \le 200) - P(X \le 100) = .9375 - .75 = .1875$

2.3 $X \sim \text{Exp}(\lambda)$

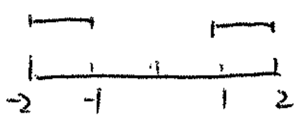
$F(x) = 1 - e^{-\lambda x} = \frac{1}{2} \Rightarrow e^{-\lambda x} = \frac{1}{2} \Rightarrow -\lambda x = \ln\frac{1}{2} \Rightarrow \boxed{x = \frac{1}{\lambda}\ln 2}$

2.4. $f(t) = \frac{1}{4}t^2 + Yt + 1$

$\Delta = Y^2 - 1$

$P(Y^2 - 1 \ge 0) = P(Y^2 \ge 1) = P(Y \ge 1) + P(Y \le -1) = \boxed{\frac{1}{2}}$

$Y \sim \text{Uniform}(-2, 2)$



2.5. $X \sim Exp(\lambda)$

$Y = k$ if $k-1 \le X < k$ $\quad k = 1, 2, 3, \cdots$

$$P(Y=k) = P(k-1 \le X < k) = F(k) - F(k-1) = (1-e^{-\lambda k}) - (1-e^{-\lambda(k-1)})$$
$$= e^{-\lambda k} e^{\lambda} - e^{-\lambda k}$$
$$= e^{-\lambda k}(e^{\lambda} - 1)$$

For a geometric R.V. $f(k) = (1-p)^{k-1} p$
$$= (1-p)^k \frac{p}{1-p}$$

if let $p = 1 - e^{-\lambda}$, $P(Y=k) = (1-p)^k \cdot \frac{p}{1-p}$

hence $Y \sim \text{Geometric}(1 - e^{-\lambda})$

2.9 $\Gamma(\frac{1}{2}) = \int_0^{\infty} t^{-\frac{1}{2}} e^{-t} \, dt$

we substitude $t = \frac{y^2}{2}$. so that $dt = y\,dy = \sqrt{2t}\, dy$

hence our integral above equals $\int_0^{\infty} \sqrt{2} e^{-\frac{y^2}{2}} dy = \sqrt{\pi} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}_{\text{standard normal density}} dy$

$$= \sqrt{\pi} \cdot 1$$
$$= \sqrt{\pi}$$

2.11 ① $P(X > .25) = 1 - P(X \le .25)$

$$= 1 - \Phi\left(\frac{.25 - .30}{.06}\right) = \boxed{0.7967}$$

$X \sim \text{Normal}(.30, .06^2)$

② $P(X \le .10) = \Phi\left(\frac{.10 - .30}{.06}\right) = \boxed{0.0004}$

③ $P(X \ge x) = 1 - P(X < x) = 1 - \Phi\left(\frac{x - .30}{.06}\right) = .05$

$\Phi\left(\frac{x - .30}{.06}\right) = .95$

$\Rightarrow \quad x = .30 + (1.645) \times (.06) = \boxed{.3987}$

2.12 $X \sim \text{Normal}(\mu, \sigma^2)$

$\left(\frac{X - \mu}{\sigma}\right) \sim \text{Normal}(0, 1)$

$\left(\frac{X - \mu}{\sigma}\right)^2 \sim \chi^2_1$