550.310 HW5 - solution

1.2 X = "net gain"

$$E(X) = (-.5)\cdot\frac{3{,}000{,}000-12{,}006}{3{,}000{,}000} + (25-.5)\cdot\frac{12{,}000}{3{,}000{,}000}$$

$$+ (10{,}000-.5)\cdot\frac{4}{3{,}000{,}000} + (50{,}000-.5)\cdot\frac{1}{3{,}000{,}000}$$

$$+ (200{,}000-.5)\cdot\frac{1}{3{,}000{,}000} \approx \boxed{-0.3033}$$ (50 cents per ticket)

1.3 ① $E(Y) = \sum x_i f(x_i) = 0\times\frac{1}{2} + 1\times\frac{3}{8} + 2\times\frac{1}{8} = \frac{5}{8}$

② pmf of Y: $P(Y=0) = \frac{1}{2}$

$P(Y=1) = \frac{3}{8}$

$P(Y=4) = \frac{1}{8}$

$E(Y) = \sum y_i f(y_i) = 0\times\frac{1}{2} + 1\times\frac{3}{8} + 4\times\frac{1}{8} = \frac{7}{8}$

③ $E(X^2) = \sum x_i^2 f(x_i) = 0\times\frac{1}{2} + 1^2\times\frac{3}{8} + 2^2\times\frac{1}{8} = \frac{7}{8} = E(Y)$

④ $Var(X) = E(X^2) - E^2(X) = \frac{7}{8} - \left(\frac{5}{8}\right)^2 = \frac{31}{64}$

1.4. $E(X) = \sum_{k=1}^{\infty} k\cdot\frac{6}{\pi^2 k^2} = \frac{6}{\pi^2}\cdot\sum_{k=1}^{\infty}\frac{1}{k}$

$\sum_{k=1}^{\infty}\frac{1}{k}$ is harmonic series, it diverges.

Thus $E(X)$ doesn't exist.

1.16. $X \sim \text{Poisson}(1.6)$. We need to find the smallest k such that

$P(X \le k) \ge 85\%$

x	0	1	2	$\boxed{3}$
$P(X=k)$	.20	.32	.26	.14
$P(X \le k)$	.20	.52	.78	.92

1.18 $p = \frac{30}{3714}$, $n = 500$

X = " # of positive among the 500"

$X \sim \text{Binomial}(500, \frac{30}{3714}) \simeq \text{Poisson}(4.001)$

$P(X=10) = e^{-4.001} \cdot \frac{(4.001)^{10}}{10!} \simeq \boxed{0.005}$

1.19 $P = \frac{12}{1000}$, $n = 100$

$X \sim \text{Binomial}(100, \frac{12}{1000}) \simeq \text{Poisson}(1.2)$

$P(X \ge 4) = 1 - e^{-1.2}[1 + 1.2 + \frac{(1.2)^2}{2} + \frac{(1.2)^3}{6}] \approx \boxed{.03}$

2.1 ① $\int_0^1 cx^2 dx = 1$

$\Rightarrow \frac{c}{3}x^3\Big|_0^1 = 1$

$\Rightarrow \boxed{c=3}$

② cdf of X:

$$F(a) = \begin{cases} 0 & a < 0 \\ \int_0^a 3x^2 dx = a^3 & 0 \le a \le 1 \\ 1 & a > 1 \end{cases}$$

③ $P(.1 \le X \le .5) = F(.5) - F(.1) = \boxed{0.124}$

3.5 ① for fixed α

$$E(X) = \int_{-1}^{1} x \cdot \frac{1+\alpha x}{2} \cdot dx = \int_{-1}^{1} \frac{1}{2}x + \frac{\alpha}{2} \cdot x^2 \, dx$$

$$= \cancel{\frac{1}{4}x^2\Big|_{-1}^{1}} + \frac{\alpha}{6} \cdot x^3\Big|_{-1}^{1}$$

$$= \boxed{\frac{\alpha}{3}}$$

$$E(X^2) = \int_{-1}^{1} x^2 \cdot \frac{1+\alpha x}{2} \cdot dx = \int_{-1}^{1} \frac{1}{2}x^2 + \frac{\alpha}{2} \cdot x^3 \, dx$$

$$= \frac{1}{6}x^3\Big|_{-1}^{1} + \cancel{\frac{\alpha}{8} \cdot x^4\Big|_{-1}^{1}}$$

$$= \frac{1}{3}$$

thus $\sigma_X^2 = E(X^2) - E^2(X) = \frac{1}{3} - \left(\frac{\alpha}{3}\right)^2 = \boxed{\frac{3-\alpha^2}{9}}$ ③