550.310

HW4 solution

1.1 $f_M(1) = P(M=1) = P\{(1,1)\} = \frac{1}{36}$

$f_M(2) = P(M=2) = P\{(1,2), (2,1), (2,2)\} = \frac{3}{36} = \frac{1}{12}$

$f_M(3) = P(M=3) = P\{(1,3), (2,3), (3,3), (3,1), (3,2)\} = \frac{5}{36}$

$f_M(4) = P(M=4) = P\{(1,4), (2,4), (3,4), (4,4), (4,1), (4,2), (4,3)\} = \frac{7}{36}$

$f_M(5) = P(M=5) = P\{(1,5), (2,5), (3,5), (4,5), (5,5), (5,1), (5,2), (5,3), (5,4)\} = \frac{9}{36} = \frac{1}{4}$

$f_M(6) = P(M=6) = P\{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\} = \frac{11}{36}$

$$E[M] = 1\cdot f_M(1) + 2\cdot f_M(2) + 3\cdot f_M(3) + 4\cdot f_M(4) + 5\cdot f_M(5) + 6\cdot f_M(6)$$

$$= 1\cdot\frac{1}{36} + 2\cdot\frac{1}{12} + 3\cdot\frac{5}{36} + 4\cdot\frac{7}{36} + 5\cdot\frac{1}{4} + 6\cdot\frac{11}{36}$$

$$= \frac{1+6+15+28+45+66}{36} = \frac{161}{36} \approx 4.47$$

1.5 X = "Number of children that inherit the desease"

$X \sim Bin(5, \frac{1}{4})$

$$P(2 \le X \le 4) = P(X=2) + P(X=3) + P(X=4)$$

$$= \binom{5}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 + \binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 + \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1$$

$$\approx 0.36$$

1.6 $X =$ " Number of white blood cells that are neutrophils "

$X \sim Bin(8, 0.4)$

$P(X=3) = \binom{8}{3}(0.4)^3(0.6)^5 \approx 0.28$

1.8 7 bit word :

$P(\text{"correctly received"}) = P(0 \text{ error}) + P(1 \text{ error})$

$= \binom{7}{0}(0.05)^0(0.95)^7 + \binom{7}{1}(0.05)^1(0.95)^6$

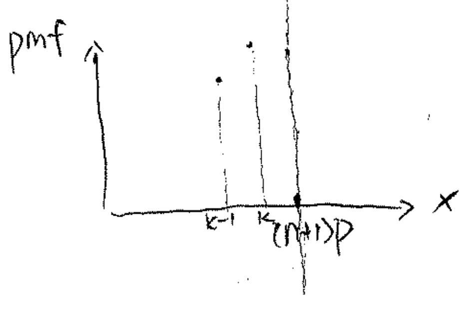
≈ 0.96

4 bit word :

$P(\text{" correctly received"}) = P(0 \text{ error}) = (0.95)^4 \approx 0.81$

1.10 Consider the ratio $\dfrac{P(X=k)}{P(X=k-1)}$, set the ratio to 1 .

$$\frac{\binom{n}{k}p^k(1-p)^{n-k}}{\binom{n}{k-1}p^{k-1}(1-p)^{n-k+1}} = \frac{\frac{\cancel{n!}}{k!(n-k)!}p^k(1-p)^{n-k}}{\frac{\cancel{n!}}{(k-1)!(n-k+1)!}p^{k-1}(1-p)^{n-k+1}} = \frac{n-k+1}{k}\cdot\frac{p}{1-p} = 1$$



$(n-k+1)p = k(1-p)$

$np - \cancel{kp} + p = k - \cancel{kp}$

$k = (n+1)p$

when $k \le (n+1)p$. $P(X=k) \ge P(X=k-1)$

when $k > (n+1)p$. $P(X=k) \le P(X=k-1)$

therefore $k = \lfloor (n+1)p \rfloor$

(" $\lfloor \ \rfloor$ " denotes the largest integer less than $(n+1)p$)

1.14 For any nonnegative integer m,

$$P(X>m) = 1-P(X\le m) = 1-\sum_{i=1}^{m} p(1-p)^{i-1} = 1-p\cdot\frac{1-(1-p)^m}{1-(1-p)} = (1-p)^m$$

$$P(X>n+k-1 \mid X>n-1) = \frac{P(X>n+k-1)}{P(X>n-1)} = \frac{(1-p)^{n+k-1}}{(1-p)^{n-1}} = (1-p)^k = P(X>k)$$

It is "obvious" since the probability of not getting a head in the next k flips is unaffected by whether you are starting from scratch or whether you are starting after having already flipped $n-1$ tails.

1.15 $X =$ "Number of simultaneous throws until all of them show the same face"

$X \sim \text{Geometric}(\frac{1}{4})$

$$p = P(\text{all coins are the same face}) = P(HHH) + P(TTT) = (\tfrac{1}{2})^3 + (\tfrac{1}{2})^3 = \tfrac{1}{4}$$

$$P(3\le X\le 5) = P(X=3)+P(X=4)+P(X=5) = (\tfrac{3}{4})^2(\tfrac{1}{4}) + (\tfrac{3}{4})^3(\tfrac{1}{4}) + (\tfrac{3}{4})^4(\tfrac{1}{4})$$

$$\approx 0.325$$

1.20 Similar to problem 1.10, consider the ratio $\frac{P(X=k)}{P(X=k-1)}$, set it to 1

$$\frac{e^{-s}\frac{s^k}{k!}}{e^{-s}\cdot\frac{s^{k-1}}{(k-1)!}} = \frac{s}{k} = 1 \qquad k=s$$

when $k\le s$, $P(X=k)\ge P(X=k-1)$

when $k>s$, $P(X=k)\le P(X=k-1)$

therefore, when $k=\lfloor s\rfloor$, $P(X=k)$ is maximized.

1.21 $P(X=k) = e^{-s}\frac{s^k}{k!}$

take derivative of $P(X=k)$ with respect to s, set the derivative to 0.

$$\frac{1}{k!}(e^{-s}k\,s^{k-1} - e^{-s}s^k) = 0$$

$$\frac{1}{k!}e^{-s}(ks^{k-1} - s^k) = 0$$

$$ks^{k-1} - s^k = 0$$

$$s = k$$

therefore, when $s=k$, $P(X=k)$ is maximized.