550.310

HW 11 - solution

17.1. a) $\bar{x} = 9.81$ $S = 14.41$ $n = 426$

95% confidence interval $\Rightarrow 1-\alpha = 0.95 \Rightarrow \alpha = 0.05$

$\frac{\sqrt{n}}{S}(\bar{x}-\mu) \sim$ student $T(n-1)$

The 95% C.I. for μ is $\boxed{\bar{x} \pm \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2)}$

$9.81 \pm \frac{14.41}{\sqrt{426}} \cdot t_{425}(0.025)$

$9.81 \pm \frac{14.41}{\sqrt{426}} \cdot 1.96$

$\boxed{9.81 \pm 1.37}$

b) $0.5 = \frac{14.41}{\sqrt{n}} t_{425}(0.025) = \frac{14.41}{\sqrt{n}} \cdot 1.96$

$\boxed{n = 3191}$

17.4 PAIS: $n = 102$ $S = 16.08$ $\bar{x} = 36.5$

90% confidence interval $\Rightarrow 1-\alpha = 0.9 \Rightarrow \alpha = 0.1$

The 90% C.I. for μ is $\boxed{\bar{x} \pm \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2)}$

$36.5 \pm \frac{16.08}{\sqrt{102}} \cdot t_{101}(0.05)$

$36.5 \pm \frac{16.08}{\sqrt{102}} \cdot 1.66$

$\boxed{36.5 \pm 2.64}$

Serum phosphate : $n=102$, $s=0.47$, $\bar{x}=1.68$, $\alpha=0.1$

The 90% C.I. for μ is $\boxed{\bar{x} \pm \frac{s}{\sqrt{n}} t_{n-1}(\alpha/2)}$

$$1.68 \pm \frac{0.47}{\sqrt{102}} \cdot t_{101}(0.05)$$

$$1.68 \pm \frac{0.47}{\sqrt{102}} \cdot 1.66$$

$$\boxed{1.68 \pm 0.077}$$

17.8 $\bar{x} = \frac{2.84+3.54+2.8+1.44+2.94+2.7}{6} = 2.71$ $n=6$.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = 0.476$$

$s = 0.69$

95% confidence interval $\Rightarrow \alpha = 0.05$

The 95% C.I. for μ is $\boxed{\bar{x} \pm \frac{s}{\sqrt{n}} t_{n-1}(\alpha/2)}$

$$2.71 \pm \frac{0.69}{\sqrt{6}} \cdot t_5(0.025)$$

$$2.71 \pm \frac{0.69}{\sqrt{6}} \cdot 2.571$$

$$\boxed{2.71 \pm 0.72}$$

16.1 a) $\hat{p} = \frac{381}{1000} = 0.381$, $n = 1000$, 95% C.I. $\Rightarrow \alpha = 0.05$.

The 95% C.I. for p is $\boxed{\hat{p} \pm \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \cdot z(\alpha/2)}$

$$0.381 \pm \sqrt{\frac{0.381 \times 0.619}{1000}} \cdot z(0.025)$$

$$0.381 \pm \sqrt{\frac{0.381 \times 0.619}{1000}} \cdot 1.96$$

$$\boxed{0.381 \pm 0.03}$$

b) $0.15 = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \cdot z(0.025) = \sqrt{\frac{0.4 \times 0.6}{1000}} \cdot 1.96$

$\boxed{n = 41}$

16.3 $\hat{p} = \frac{22}{110} = 0.2$ $n = 110$ 90% C.I. $\Rightarrow \alpha = 0.1$

The 90% C.I. for p is $\boxed{\hat{p} \pm \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \cdot z(\alpha/2)}$

$$0.2 \pm \sqrt{\frac{0.2 \times 0.8}{110}} \, z(0.05)$$

$$0.2 \pm \sqrt{\frac{0.2 \times 0.8}{110}} \cdot 1.645$$

$$\boxed{0.2 \pm 0.063}$$

16.4 X = "# of C.I.s including θ"

P(one single 95% C.I. includes θ) $= 0.95$

$n=1000$

$X \sim \text{Binomial}(1000, 0.95)$

$\approx \text{Normal}(1000\times 0.95,\ 1000\times 0.95\times 0.05) = \text{Normal}(950, 47.5)$

so $P(940 \le X \le 960) = P\left(\frac{940-950}{\sqrt{47.5}} \le \frac{X-950}{\sqrt{47.5}} \le \frac{960-950}{\sqrt{47.5}}\right)$

$= \Phi\left(\frac{960-950}{\sqrt{47.5}}\right) - \Phi\left(\frac{940-950}{\sqrt{47.5}}\right)$

$= \Phi(1.45) - \Phi(-1.45)$

$= 2\Phi(1.45) - 1$

$= 2\times 0.9265 - 1 = \boxed{0.853}$

16.5 $\sigma_{\hat{p}}^2 = \text{Var}(\hat{p}) = \frac{p(1-p)}{n}$

$\boxed{\text{Want to show } E\left(\frac{\hat{p}(1-\hat{p})}{n-1}\right) = \frac{p(1-p)}{n}}$

we know that $E(\hat{p}) = p$, $\text{Var}(\hat{p}) = \frac{p(1-p)}{n} = E(\hat{p}^2) - [E(\hat{p})]^2$

so $E(\hat{p}^2) = \frac{p(1-p)}{n} + p^2$

so $E\left(\frac{\hat{p}(1-\hat{p})}{n-1}\right) = \frac{E(\hat{p})}{n-1} - \frac{E(\hat{p}^2)}{n-1} = \frac{p}{n-1} - \left(\frac{p(1-p)}{n} + p^2\right)\cdot\frac{1}{n-1}$

$$= \frac{np - p(1-p) - np^2}{n(n-1)} = \frac{np(1-p) - p(1-p)}{n(n-1)}$$

$$= \frac{p(1-p)\cancel{(n-1)}}{n\cancel{(n-1)}}$$

$$= \frac{p(1-p)}{n}$$

Q.E.D

Chapter 7:

58. a). $P(\min(X_i) \le \tilde{\mu} \le \max(X_i))$

$= 1 - P(\min(X_i) > \tilde{\mu} \cup \tilde{\mu} > \max(X_i))$

$= 1 - P(\tilde{\mu} < \min(X_i)) - P(\tilde{\mu} > \max(X_i))$

$= 1 - P(\tilde{\mu} < X_1, \tilde{\mu} < X_2, \cdots \tilde{\mu} < X_n) - P(\tilde{\mu} > X_1, \tilde{\mu} > X_2, \cdots \tilde{\mu} > X_n)$

X_i are iid $= 1 - [P(\tilde{\mu} < X_1)]^n - [P(\tilde{\mu} > X_1)]^n$

$= 1 - (\frac{1}{2})^n - (\frac{1}{2})^n$

$= 1 - (\frac{1}{2})^{n-1}$

b). By part a), $(\min(X_i), \max(X_i))$ is a $(1-\alpha)$ confidence interval for $\tilde{\mu}$, and $1-\alpha = 1 - (\frac{1}{2})^{n-1}$

when $n=6$, $1-\alpha = 1-(\frac{1}{2})^5 = 1 - \frac{1}{32} \approx 0.97$

so we can use the result of a). a 97% C.I. for $\tilde{\mu}$ is $\boxed{(1.44, 3.54)}$.

(5)

c). $P(X_{(2)} < \tilde{\mu} < X_{(n-1)}) = 1 - P(\tilde{\mu} \leq X_{(2)}) - P(\tilde{\mu} \geq X_{(n-1)})$

$= 1 - P($at most one x_i is less than $\tilde{\mu}) - P($at most one is greater than $\tilde{\mu})$

$= 1 - \underline{(\frac{1}{2})^n} - \binom{n}{1}(\frac{1}{2})(\frac{1}{2})^{n-1} - \underline{(\frac{1}{2})^n} - \binom{n}{1}(\frac{1}{2})(\frac{1}{2})^{n-1}$

$= 1 - (\frac{1}{2})^{n-1} - \cancel{2}n(\cancel{\tfrac{1}{2}})(\frac{1}{2})^{n-1}$

$= \underbrace{1 - (n+1)(\frac{1}{2})^{n-1}}_{\text{confidence coefficient}}$

59. a). $P((\alpha/2)^{1/n} < \frac{Y}{\theta} \leq (1-\alpha/2)^{1/n}) = 1-\alpha$

$$\int_{(\alpha/2)^{1/n}}^{(1-\alpha/2)^{1/n}} n u^{n-1}\,du = u^n \Big|_{(\alpha/2)^{1/n}}^{(1-\alpha/2)^{1/n}} = [(1-\alpha/2)^{\cancel{1/n}}]^{\cancel{n}} - [(\alpha/2)^{\cancel{1/n}}]^{\cancel{n}}$$

$$= 1 - \alpha/2 - \alpha/2 = 1-\alpha$$

$$P\left(\frac{(\alpha/2)^{1/n}}{Y} < \frac{1}{\theta} \leq \frac{(1-\alpha/2)^{1/n}}{Y}\right) = 1-\alpha$$

$$P\left(\frac{Y}{(1-\alpha/2)^{1/n}} \leq \theta < \frac{Y}{(\alpha/2)^{1/n}}\right) = 1-\alpha$$

so $\boxed{\left(\frac{\max(X_i)}{(1-\alpha/2)^{1/n}}, \frac{\max(X_i)}{(\alpha/2)^{1/n}}\right)}$ is a $100(1-\alpha)\%$ CI for θ.

b) $P(\alpha^{1/n} \le Y/\theta \le 1) = 1-\alpha$

$$\int_{\alpha^{1/n}}^{1} n u^{n-1}\, du = u^n \Big|_{\alpha^{1/n}}^{1} = 1^n - [(\alpha)^{1/n}]^n = 1-\alpha$$

$$P\left(\frac{\alpha^{1/n}}{Y} \le \frac{1}{\theta} \le \frac{1}{Y}\right) = 1-\alpha$$

$$P\left(Y \le \theta \le \frac{Y}{\alpha^{1/n}}\right) = 1-\alpha$$

so $\left(\max(X_i), \frac{\max(X_i)}{\alpha^{1/n}}\right)$ is a $100(1-\alpha)\%$ CI for θ

c). The interval in part b is shorter.

$95\% \Rightarrow \alpha = 0.05$

$\max(X_i) = 4.2$

$n=5$

$$\left(4.2, \frac{4.2}{0.05^{1/5}}\right) = (4.2, 7.65)$$

60. To minimize w is to minimize $z_r + z_{\alpha-r}$

$$\Phi(z_\alpha) = 1-\alpha, \quad z_\alpha = \Phi^{-1}(1-\alpha)$$

so $z_r + z_{\alpha-r} = \Phi^{-1}(1-r) + \Phi^{-1}(1-(\alpha-r))$

set $\frac{d}{dr}\left(\Phi^{-1}(1-r) + \Phi^{-1}(1-(\alpha-r))\right) = 0$

$$\frac{d\Phi^{-1}(1-r)}{dr} + \frac{d\Phi^{-1}(1-\alpha+r)}{dr}$$

$$= \frac{d\Phi^{-1}(1-r)}{d(1-r)} \cdot \frac{d(1-r)}{dr} + \frac{d\Phi^{-1}(1-\alpha+r)}{d(1-\alpha+r)} \cdot \frac{d(1-\alpha+r)}{dr}$$

$$= -\frac{d\Phi^{-1}(1-r)}{d(1-r)} + \frac{d\Phi^{-1}(1-\alpha+r)}{d(1-\alpha+r)}$$

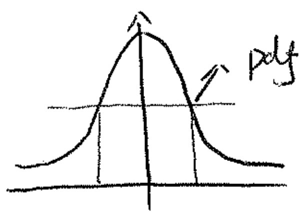
think of $(1-r)$ as y

$$x = f^{-1}(y) = \Phi^{-1}(1-r) = Z_r$$

$$\frac{df^{-1}(y)}{dy} = \frac{1}{f'(x)}$$

$$= -\frac{1}{\Phi'(Z_r)} + \frac{1}{\Phi'(Z_{\alpha-r})}$$

$$\Phi'(Z_r) = \Phi'(Z_{\alpha-r})$$



$$\varphi(Z_r) = \varphi(Z_{\alpha-r})$$

$\Rightarrow$ $Z_r = Z_{\alpha-r}$ or

$Z_r = -Z_{\alpha-r}$

but Z_α always > 0

so $Z_r = Z_{\alpha-r}$

so $r = \alpha - r$

$\Rightarrow$ $r = \frac{\alpha}{2}$